A discrete curvature approach to strongly spherical graphs

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Guangzhou Discrete Math Seminar @SYSU

December 14, 2018

joint work with

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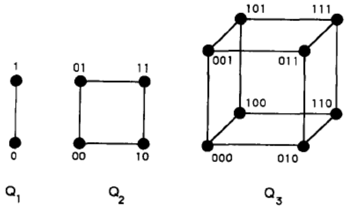
Combinatorial intuitions

Hypercubes

The *n-dimensional hypercube* Q_n is defined recursively in terms of Cartesian product of two graphs: [1]

$$Q_1 = K_2,$$
$$Q_n = K_2 \times Q_{n-1}.$$



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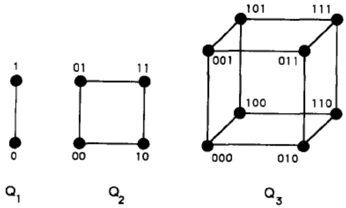
$$Q_n = K_2 \times Q_{n-1}.$$

- Vertex: 2^n n-dim boolean vectors;
- Edges: Two vertices are adjacent whenever they differ in exactly one coordinate.



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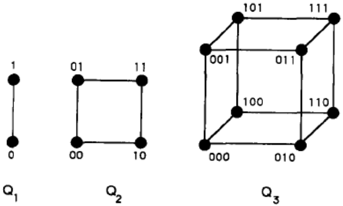
Analogies between Spheres and Hypercubes



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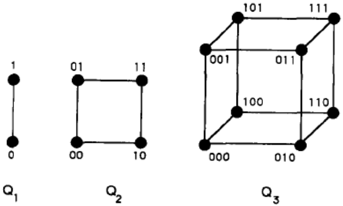

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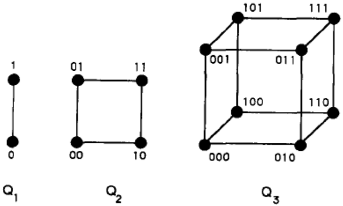
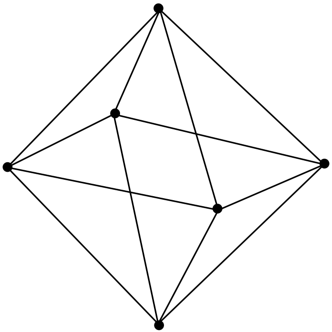


- Every point has an antipodal point.
- For every two distinct x, y, all the geodesics connecting x, y run over a (low-dim) hypercube.

More candidates?



- For every x, we can find a $\bar{x}$ such that $[x, \bar{x}] = V$(antipodal). [2]
- For every pair $x, y \in V$, $x \neq y$, $[x, y]$ is again antipodal.[3]

[2] The interval between x and y is the subset of V given by

$$[x, y] = \{z \in V : d(x, y) = d(x, z) + d(z, y)\}.$$

[3] For simplicity, we also use $[x, y]$ for the subgraph induced by the interval.

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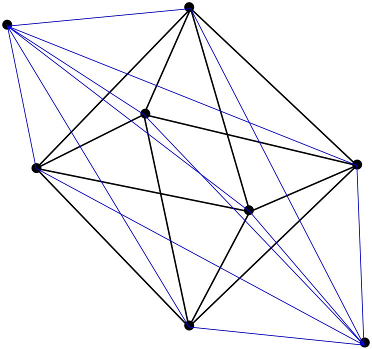

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Spherical graphs

Spherical graphs were introduced by Berrachedi, Havel, Mulder in 2003[4] and represent an interesting generalization of hypercubes.

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- We call a connected graph $G = (V, E)$ antipodal if for every vertex $x \in V$ there exists some vertex $y \in V$ with $[x, y] = V$.

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- We call a connected graph $G = (V, E)$ spherical if each of its interval is antipodal.

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- We call a connected graph $G = (V, E)$ strongly spherical if it is both antipodal and spherical.

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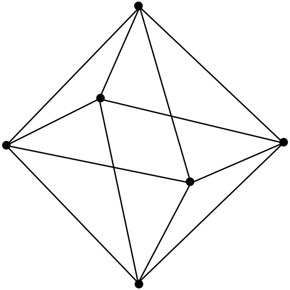
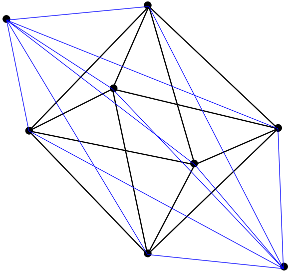
More Examples





- Cocktail party graphs $CP(n)$ obtained by removal of a perfect matching from the complete graph K_{2n};
- Johnson graphs $J(2n, n)$ with vertices corresponding to n-subsets of $\{1, 2, \cdots, 2n\}$ and edges between them if they overlap in $n - 1$ elements;
- Even-dimensional demi-cubes $Q^{2n}_{(2)}$: one of the two isomorphic connected components of the vertex set $\{0, 1\}^{2n}$ and edges between them if Hamming distance equals two;

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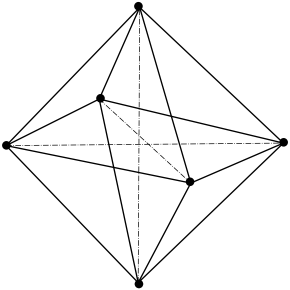

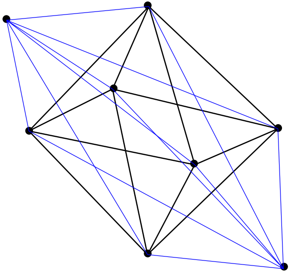

- Cocktail party graphs $CP(n)$ obtained by removal of a perfect matching from the complete graph K_{2n}; $CP(3)$!!
- Johnson graphs $J(2n, n)$ with vertices corresponding to n-subsets of $\{1, 2, \cdots, 2n\}$ and edges between them if they overlap in $n-1$ elements;
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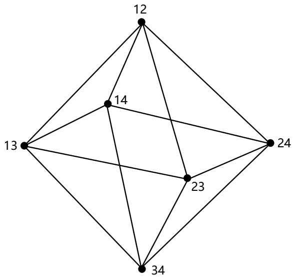

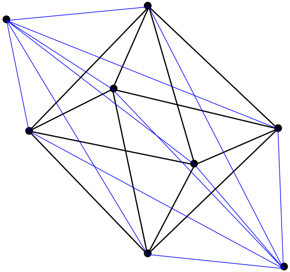

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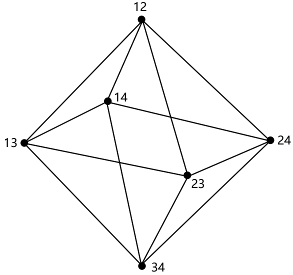

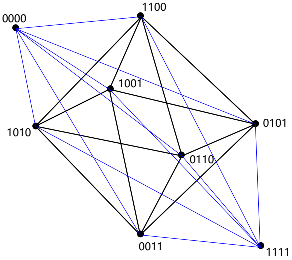

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- Even-dimensional demi-cubes $Q^{2n}_{(2)}$: one of the two isomorphic connected components of the vertex set $\{0, 1\}^{2n}$ and edges between them if Hamming distance equals two; $Q^{4}_{(2)}$!!

Classification of strongly spherical graphs

Theorem (Koolen-Moulton-Stevanović 2004)

Strongly spherical graphs are precisely the Cartesian products

$$G_1 \times G_2 \times \cdots \times G_k,$$

where each factor G_i is either

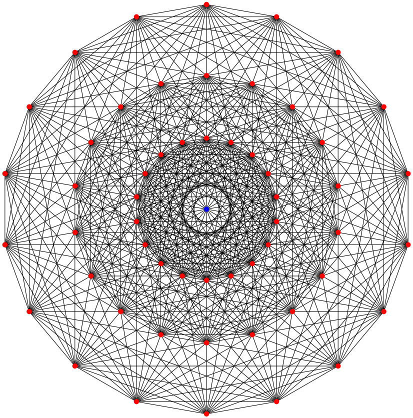
- *a hypercube*
- *a cocktail party graph*
- *a Johnson graph $J(2n, n)$*
- *an even dimensional demi-cube*
- *or the Gosset graph.* [5]

---

[5] A Gosset graph has 56 vertices:

- the vertices are in one-one correspondence with the edges $\{i, j\}$ and $\{i, j\}'$ of two disjoint copies of K_8, respectively.
- $\{i, j\} \sim \{k, l\}$ if $|\{i, j\} \cap \{k, l\}| = 1$ and $\{i, j\} \sim \{k, l\}'$ if $\{i, j\} \cap \{k, l\} = \emptyset$.

Gosset graph



A characterization of spheres in Riemannian geometry

Bonnet-Myers and Cheng Theorems

Theorem (Bonnet 1855; Myers 1941 Duke Math. J.)

Let (M, g) be a complete Riemannian manifold with $\mathrm{Ric} \geq (n-1)k$. Then we have M is compact and

$$\mathrm{diam}(M, g) \leq \frac{\pi}{\sqrt{k}}.$$

Theorem (Cheng 1975)

Let (M, g) be a complete Riemannian manifold with $\mathrm{Ric} \geq (n-1)k$. Then we have

$$\mathrm{diam}(M, g) = \frac{\pi}{\sqrt{k}}$$

if and only if M is the sphere $S^n(\frac{1}{\sqrt{k}})$.

Lichnerowicz and Obata Theorems

Theorem (Lichnerowicz 1958)

Let (M, g) be a complete Riemannian manifold with $\mathrm{Ric} \geq (n-1)k$. Then we have the smallest positive Laplace-Beltrami eigenvalue satisfies

$$\lambda_1(M, g) \geq nk.$$

Theorem (Obata 1962)

Let (M, g) be a complete Riemannian manifold with $\mathrm{Ric} \geq (n-1)k$. Then we have

$$\lambda_1(M, g) = nk$$

if and only if M is the sphere $S^n(\frac{1}{\sqrt{k}})$.

Question: Discrete Analogues?

Discrete setting

- $G = (V, E)$: V is a countable set.
- Locally finite: $\mathrm{Deg}(x) := \sharp\{y \in V | y \sim x\} < \infty, \forall x \in V$
- For any $f : V \to \mathbb{R}$, $x \in V$, consider the Laplacian Δ:

$$\Delta f(x) := \frac{1}{\mathrm{Deg}(x)} \sum_{y, y \sim x} (f(y) - f(x)).$$

Ollivier-Ricci curvature

Ollivier-Ricci curvature $\kappa(x, y)$ is a notion based on optimal transport and is defined on pairs of different vertices $x, y \in V$.

Intuition: $\kappa(x, y) > 0$ if the average distance between corresponding neighbours of x and y is smaller than $d(x, y)$.

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We represent the neighbours of x by the following probability measures μ_x^p for any $x \in V$, $p \in [0, 1]$:

$$\mu_x^p(z) = \begin{cases} p & \text{if } z = x, \\ \frac{1-p}{\mathrm{Deg}(x)} & \text{if } z \sim x, \\ 0 & \text{otherwise.} \end{cases}$$

Wasserstein distance

Definition

Let $G = (V, E)$ be a graph. Let μ_1, μ_2 be two probability measures on V. The *Wasserstein distance* $W_1(\mu_1, \mu_2)$ between μ_1 and μ_2 is defined as

$$W_1(\mu_1, \mu_2) := \inf_{\pi \in \Pi(\mu_1, \mu_2)} \sum_{x \in V} \sum_{y \in V} d(x, y)\pi(x, y),$$

where π runs over all transport plans in

$$\Pi(\mu_1, \mu_2) = \left\{ \pi : V \times V \to [0, 1] : \mu_1(x) = \sum_{y \in V} \pi(x, y),\ \mu_2(y) = \sum_{x \in V} \pi(x, y) \right\}$$

Ollivier-Ricci curvature

Definition (Ollivier 2009)

Let $p \in [0, 1]$. The *p-Ollivier Ricci curvature* between two different vertices $x, y \in V$ is

$$\kappa_p(x, y) = 1 - \frac{W_1(\mu_x^p, \mu_y^p)}{d(x, y)},$$

where p is called the *idleness*.

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Definition (Lin-Lu-Yau 2011)

The Lin-Lu-Yau curvature between two neighboring vertices $x \sim y$ is

$$\kappa(x,y) := \kappa_{LLY}(x,y) = \lim_{p \to 1} \frac{\kappa_p(x,y)}{1-p}.$$

Discrete Bonnet-Myers theorem

Theorem (Ollivier '09, Lin-Lu-Yau '11)

Let $G = (V, E)$ *be a connected graph and* $\inf_{x \sim y} \kappa(x, y) > 0$*. Then* G *has finite diameter* $L := \mathrm{diam}(G) < \infty$ *and*

$$\inf_{x \sim y} \kappa(x, y) \leq \frac{2}{L}.$$

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We say that a D-regular graph G with diameter L is ***(D, L)-Bonnet-Myers sharp*** if the inequality holds with equality.

Discrete Lichnerowicz theorem

Theorem (Ollivier '09, Lin-Lu-Yau '11)

Let $G = (V, E)$ be a finite connected graph. Then we have for the smallest positive solution λ_1 of $\Delta f + \lambda_1 f = 0$

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Relations between these two classes of graphs

Theorem (Cushing-Kamtue-Koolen-L.-Münch-Peyerimhoff)

Any Bonnet-Myers sharp graph is Lichnerowicz sharp.

Basic Properties

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Any (D, L)-Bonnet-Myers sharp graph satisfies $L \leq D$. Moreover L must divide $2D$.

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Theorem (CKKLMP)

$G_1 \times G_2 \times \cdots \times G_k$ is Bonnet-Myers sharp if and only if all factors G_i are Bonnet-Myers sharp and satisfy

$$\frac{D_1}{L_1} = \frac{D_2}{L_2} = \cdots = \frac{D_k}{L_k}.$$

Discrete Cheng Theorem

We can classify all self-centered[6] Bonnet-Myers sharp graphs:

Theorem (CKKLMP)

Self-centered Bonnet-Myers sharp graphs are precisely the following graphs:

1. *hypercubes* Q^n
2. *cocktail party graphs* $CP(n)$
3. *the Johnson graphs* $J(2n, n)$
4. *even-dimensional demi-cubes* $Q^{2n}_{(2)}$
5. *the Gosset graph*

and Cartesian products of 1.-5. satisfying the condition $D_i/L_i = const$.

[6] a graph $G = (V, E)$ is called self-centered if, for every vertex $x \in V$, there exists a vertex $\overline{x} \in V$ such that $d(x, \overline{x}) = \mathrm{diam}(G)$.

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In fact, we show that every self-centered Bonnet-Myers sharp graph is strongly spherical!!

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A combinatorial description

Definition

Let $G = (V, E)$ be a regular graph. We say G satisfies $\Lambda(m)$ at an edge $e = \{x, y\} \in E$ if the following holds:

(i) e is contained in at least m triangles;

(ii) there is a perfect matching between the neighbours of x and the neighbours of y no involved in these triangles.

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Theorem (CKKLMP)

Let G be a D-regular finite connected graph of diameter L. The following are equivalent

- *G is self-centered Bonnet-Myers sharp.*
- *G is self-centered and satisfies $\Lambda(\frac{2D}{L} - 2)$.*

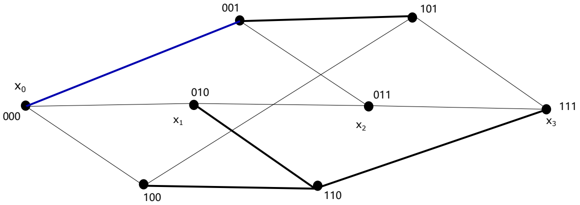
Moreover, if any of these equivalent properties holds, then every edge of G lies in precisely $\frac{2D}{L} - 2$ triangles.

A combinatorial description

Theorem (CKKLMP)

Let G be a D-regular finite connected graph of diameter L. Assume that G is self-centered and satisfies $\Lambda(\frac{2D}{L} - 2)$. Then G is strongly spherical.

Transport geodesic techniques



- Full-length geodesic: $x_0 - x_1 - x_2 - x_3$
- Transport geodesic: $000 - 000 - 001 - 101$

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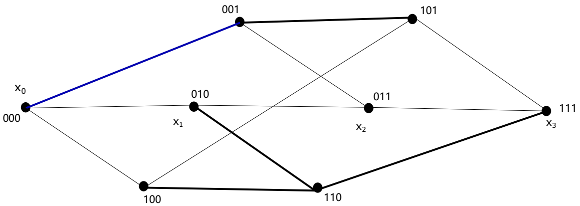

- Full-length geodesic: $x_0 - x_1 - x_2 - x_3$
- Transport geodesic: $000 - 000 - 001 - 101$
- $[x_0, x_2] = [x_1, 001]$ and $[x_0, x_3] = [x_1, 101]$

Thank you for your attention!